Some Rmk:

1. For a function with 3 variables, we also have gradient

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

§ Surfaces in $\mathbb{R}^3$ and their curvatures

Let $f(x,y) = z$ be the graph of f. For any $(x_0, y_0) \in D$, we have the tangent plane of f at $(x_0\ y_0)$ is

$$E = \left\{ \frac{\partial f}{\partial x}(x_0 y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0 y_0)(y - y_0) + f(x_0 y_0) = z \right\}$$

So, the normal vector can be choosen as

$$\vec{u}(x_0 y_0) = \left(\frac{\partial f}{\partial x}(x_0 y_0), \frac{\partial f}{\partial y}(x_0 y_0), -1\right)$$

Now, we define $\vec{N} = \frac{\vec{u}}{|\vec{u}|}$, then this will be a unit normal vector. $|\vec{N}| = 1$

Notice that $$\frac{\partial}{\partial x} \vec{N} \cdot \vec{N} = 2 \frac{\partial \vec{N}}{\partial x} \cdot \vec{N} = 0$$

so $\frac{\partial \vec{N}}{\partial x} \perp \vec{N}$

Similarly, we will have

$$\frac{\partial \vec{N}}{\partial y} \perp \vec{N}$$

Therefore, $\frac{\partial \vec{N}}{\partial x}, \frac{\partial \vec{N}}{\partial y} \in E$ (can be regarded as twe vectors on E)

Recall that: E can be generated by the following two vectors:

$$\begin{cases} V_x = \left(1 \quad 0 \quad \frac{\partial f}{\partial x}(x_0\, y_0)\right) \\ V_y = \left(0 \quad 1 \quad \frac{\partial f}{\partial y}(x_0\, y_0)\right) \end{cases}$$

So we can define the following matrix:

$$S(x_0\, y_0) := \begin{pmatrix} \frac{\partial \vec{N}}{\partial x} \cdot V_x & \frac{\partial \vec{N}}{\partial y} \cdot V_x \\ \frac{\partial \vec{N}}{\partial x} \cdot V_y & \frac{\partial \vec{N}}{\partial y} \cdot V_y \end{pmatrix}$$

This map is called Shape operator.

Proposition: $S(x_0\, y_0)$ is a symmetric matrix.

(So it is diagonalizable)

Proof:

Because $\vec{N} \cdot V_x = 0$

$$\frac{\partial}{\partial y}(\vec{N} \cdot V_x) = 0 = \frac{\partial \vec{N}}{\partial y} \cdot V_x + \vec{N} \cdot \left(\frac{\partial}{\partial y} V_x\right)$$

$$\frac{\partial}{\partial y} V_x = \left(0, 0, \frac{\partial^2 f}{\partial y \partial x}(x_0 y_0)\right)$$

$$= \left(0, 0, \frac{\partial^2 f}{\partial x \partial y}(x_0 y_0)\right) = \frac{\partial}{\partial x} V_y$$

Similarly, we have

$$0 = \frac{\partial \vec{N}}{\partial x} \cdot V_y + \vec{N} \left(\frac{\partial}{\partial x} V_y\right)$$

So $\dfrac{\partial \vec{N}}{\partial x} \cdot V_y = \dfrac{\partial \vec{N}}{\partial y} V_x$ $\square$

By this proposition, we can find a matrix M such that

(M can be a orthonormal matix)

$$M^T \cdot S_{(x_0 y_0)} M = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$$

for some $K_1 \ K_2 \in \mathbb{R}$.

We call these $K_1 \ K_2$ principle curvature of the surface

$\{ z = f(x, y) \}$

Definition:

$$K = \det(S_{(x_0 y_0)}) := \text{Gaussian curvature of } \{z = f(x,y)\} \text{ at } (x_0\, y_0).$$

$$= K_1 K_2$$

$$H = \frac{1}{2} \operatorname{tr}(S_{(x_0 y_0)}) := \text{Mean curvature of } \{z = f(x,y)\} \text{ at } (x_0\, y_0)$$

(trace of $S_{(x_0 y_0)}$)

$$= \frac{1}{2}(K_1 + K_2).$$

Example: $f(x,y) = x^2 + 2y^2$, compute K, H at $(0,0)$

$$\Rightarrow \begin{cases} V_x = (1, 0\ 2x)\big|_{x=0, y=0} = (1, 0, 0) \\ V_y = (0, 1, 4y)\big|_{x=0\ y=0} = (0, 1, 0) \end{cases}$$

$$\vec{N} = \frac{\vec{u}}{|\vec{u}|} \quad \text{with} \quad \vec{u} = (2x, 4y, -1)$$

$$\text{So } \vec{N} = \left(\frac{2x}{\sqrt{4x^2+16y^2+1}}, \frac{4y}{\sqrt{4x^2+16y^2+1}}, \frac{-1}{\sqrt{4x^2+16y^2+1}} \right)$$

$$\partial_x \vec{N} = \left(\frac{2\sqrt{4x^2+16y^2+1} - x\frac{8x}{\sqrt{4x^2+16y^2+1}}}{4x^2+16y^2+1}, \frac{-2y \cdot \frac{8x}{\sqrt{4x^2+16y^2+1}}}{4x^2+16y^2+1}, \frac{8x}{(4x^2+16y^2+1)^{3/2}} \right)$$

So $\partial_x \vec{N}(0,0) = (2, 0, 0)$

$$\partial_y \vec{N} = \left(\frac{-x\cdot(32y)}{(4x^2+16y^2+1)^{3/2}} ,\ \frac{4\sqrt{4x^2+16y^2+1} - 2y\frac{32y}{\sqrt{4x^2+16y^2+1}}}{4x^2+16y^2+1} ,\ \frac{32y}{(4x^2+16y^2+1)^{3/2}} \right)$$

So $\partial_y \vec{N}(0,0) = (0, 4, 0)$

We have

$$S_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

So $(K_1, K_2) = (2, 4)$ (sometime we will assume $K_1 \geq K_2$, but here we just choose them without ordering)

We have $\begin{cases} K = 2\cdot 4 = 8 \\ H = \frac{1}{2}(2+4) = 3 \end{cases}$

<u>Proposition</u>: In general, the surface $\{z = f(x,y)\}$ is convex or concave, then $K_1 K_2$ should be both non-positive or non-negative. If $K_1 \cdot K_2 < 0$, then (x_0, y_0) is a saddle point.

Relationship with the curvature of curves :

Let $S_{(x_0 y_0)}$ be the Shape operator.

$$M^T S_{(x_0 y_0)} M = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

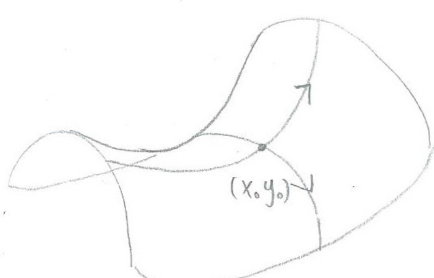
So if we take $M = (\vec{v}\ \vec{w})$, then $\begin{cases} S_{(x_0 y_0)} \vec{v} = k_1 \vec{v} \\ S_{(x_0 y_0)} \vec{w} = k_2 \vec{w} \end{cases}$

Let $\vec{v} = (v_1\ v_2)$, then it corresponds to the vector

$$(v_1 \vec{V_x} + v_2 \vec{V_y})$$

and $\vec{w} = (w_1\ w_2)$ corresponds to $(w_1 \vec{V_x} + w_2 \vec{V_y})$

We call these two directions (vectors) the principle directions.



Let $\Gamma = \{$ all smooth curves $r(t) = (x(t), y(t), z(t))$ on $\{z = f(x,y)\}$ with $r(0) = (x_0, y_0, f(x_0 y_0)).\}$

We have a map

$$\Gamma \xrightarrow{C} \mathbb{R}$$
$$r \longmapsto k(r) \quad \left(\begin{array}{c}\text{curvature of } r \\ \text{at } (x_0\ y_0)\end{array}\right)$$

<u>Proposition:</u> if $C(r_m) \leq C(r)$ for all $r \in \Gamma$ then $r_m'(0)$ is parallel to one of the principle direction; Similarly, if $C(r_M) \geq C(r)$ for all $r \in \Gamma$, then $r_M'(0)$ is parallel to the other one principle direction